

Infinite and finite dimensional generalized Hilbert tensors *

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Abstract

In this paper, we introduce the concept of an m -order n -dimensional generalized Hilbert tensor $\mathcal{H}_n = (\mathcal{H}_{i_1 i_2 \dots i_m})$,

$$\mathcal{H}_{i_1 i_2 \dots i_m} = \frac{1}{i_1 + i_2 + \dots + i_m - m + a}, \quad a \in \mathbb{R} \setminus \mathbb{Z}^-; \quad i_1, i_2, \dots, i_m = 1, 2, \dots, n,$$

and show that its H -spectral radius and its Z -spectral radius are smaller than or equal to $M(a)n^{m-1}$ and $M(a)n^{\frac{m}{2}}$, respectively, here $M(a)$ is a constant only dependent on a . Moreover, both infinite and finite dimensional generalized Hilbert tensors are positive definite for $a \geq 1$. For an m -order infinite dimensional generalized Hilbert tensor \mathcal{H}_∞ with $a > 0$, we prove that \mathcal{H}_∞ defines a bounded and positively $(m-1)$ -homogeneous operator from l^1 into l^p ($1 < p < \infty$). The upper bounds of norm of corresponding positively homogeneous operators are obtained.

Key words: Generalized Hilbert tensor, Spectral radius, Norm, Upper bounds.

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1 Introduction

Let \mathbb{R} denote the set of all real numbers, and let \mathbb{Z} be the set of all integers. We write \mathbb{Z}^- stands for the set of all non-positive integers, i.e.,

$$\mathbb{Z}^- = \{k \in \mathbb{Z} : k \leq 0\} \text{ and } \mathbb{Z}^{--} = \{k \in \mathbb{Z} : k < 0\}.$$

An m -order n -dimensional tensor (hypermatrix) $\mathcal{A} = (a_{i_1 \dots i_m})$ is a multi-array of real entries $a_{i_1 \dots i_m} \in \mathbb{R}$, where $i_j \in I_n = \{1, 2, \dots, n\}$ for $j \in I_m = \{1, 2, \dots, m\}$. \mathcal{A} is called a **sym-metric tensor** if the entries $a_{i_1 \dots i_m}$ are invariant under any permutation of their indices.

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Qi [16, 17] introduced the concepts of eigenvalues of the higher order symmetric tensors, and showed the existence of the eigenvalues and some applications. The concepts of real eigenvalue was introduced by Lim [14] independently using a variational approach. Subsequently, many mathematical workers studied the spectral properties of various structured tensors under different conditions. The spectral properties of nonnegative matrices had been generalized to n -dimensional nonnegative tensors under various conditions by Chang et al. [1, 2], He and Huang [8], He [9], He et al. [10], Li et al. [13], Qi [18], Song and Qi [21, 22], Wang et al. [25], Yang and Yang [27, 28] and references therein.

For infinite dimensional tensors, the corresponding studies have only just begun, and the conclusions are fewer. Song and Qi [20] introduced the concept of infinite dimensional Hilbert tensor and showed that such a Hilbert tensor defines a bounded, continuous and positively $(m-1)$ -homogeneous operator from l^1 into l^p ($1 < p < \infty$) and the norms of corresponding positively homogeneous operators are not larger than $\frac{\pi}{\sqrt{6}}$. They also proved that the spectral radius and E -spectral radius of finite dimensional Hilbert tensor are smaller than or equal to $n^{m-1} \sin \frac{\pi}{n}$ and $n^{\frac{m}{2}} \sin \frac{\pi}{n}$, respectively. Clearly, an m -order n -dimensional Hilbert tensor is a Hankel tensor with $v = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{nm})$, introduced by Qi [19]. Also see Chen and Qi [3], Xu [26] for more details of Hankel tensors. Hilbert tensor (hypermatrix) is a natural extension of Hilbert matrix, which was introduced by Hilbert [7]. For more details of Hilbert matrix, see Frazer[5] and Taussky [24] for n -dimensional Hilbert matrix, Choi [4]) and Ingham [11] for an infinite dimensional Hilbert matrix, Magnus [15] and Kato [12] for the spectral properties of infinite dimensional Hilbert matrix.

In this paper, we study more general Hilbert tensor, which is referred to as “generalized Hilbert tensor”. The entries of an m -order infinite dimensional generalized Hilbert tensor $\mathcal{H}_\infty = (\mathcal{H}_{i_1 i_2 \dots i_m})$ are defined by

$$\mathcal{H}_{i_1 i_2 \dots i_m} = \frac{1}{i_1 + i_2 + \dots + i_m - m + a}, \quad a \in \mathbb{R} \setminus \mathbb{Z}^-, i_1, i_2, \dots, i_m \in \mathbb{Z}^{++} = -\mathbb{Z}^{--}. \quad (1.1)$$

The entries of an m -order n -dimensional generalized Hilbert tensor $\mathcal{H}_n = (\mathcal{H}_{i_1 i_2 \dots i_m})$ are to choose $i_1, i_2, \dots, i_m \in \{1, 2, \dots, n\}$ in (1.1). Clearly, both \mathcal{H}_n and \mathcal{H}_∞ are symmetric and the entries of the generalized Hilbert tensor with $a \geq 1$ can be written as the integral form as follow

$$\mathcal{H}_{i_1 i_2 \dots i_m} = \int_0^1 t^{i_1 + i_2 + \dots + i_m - m + a - 1} dt. \quad (1.2)$$

For a vector $x = (x_1, x_2, \dots, x_n)^\top \in \mathbb{R}^n$, $\mathcal{H}_n x^{m-1}$ is a vector with its i th component defined by

$$(\mathcal{H}_n x^{m-1})_i = \sum_{i_2, \dots, i_m=1}^n \frac{x_{i_2} \dots x_{i_m}}{i + i_2 + \dots + i_m - m + a}, \quad a \in \mathbb{R} \setminus \mathbb{Z}^-, i = 1, 2, \dots, n. \quad (1.3)$$

Accordingly, $\mathcal{H}_n x^m$ is given by

$$\mathcal{H}_n x^m = x^\top (\mathcal{H}_n x^{m-1}) = \sum_{i_1, i_2, \dots, i_m=1}^n \frac{x_{i_1} x_{i_2} \dots x_{i_m}}{i_1 + i_2 + \dots + i_m - m + a}, \quad a \in \mathbb{R} \setminus \mathbb{Z}^-. \quad (1.4)$$

Let l^1 is a space consisting of all real number sequences $x = (x_i)_{i=1}^{\infty}$ satisfying

$$\sum_{i=1}^{\infty} |x_i| < \infty.$$

For a real vector $x = (x_1, x_2, \dots, x_n, x_{n+1}, \dots) \in l^1$, $\mathcal{H}_{\infty} x^{m-1}$ is an infinite dimensional vector with its i th component defined by

$$(\mathcal{H}_{\infty} x^{m-1})_i = \sum_{i_2, \dots, i_m=1}^{\infty} \frac{x_{i_2} \cdots x_{i_m}}{i + i_2 + \cdots + i_m - m + a}, \quad a \in \mathbb{R} \setminus \mathbb{Z}^-; \quad i \in \{1, 2, \dots, n, \dots\}. \quad (1.5)$$

Accordingly, $\mathcal{H}_{\infty} x^m$ is given by

$$\mathcal{H}_{\infty} x^m = \sum_{i_1, i_2, \dots, i_m=1}^{\infty} \frac{x_{i_1} x_{i_2} \cdots x_{i_m}}{i_1 + i_2 + \cdots + i_m - m + a}, \quad a \in \mathbb{R} \setminus \mathbb{Z}^-. \quad (1.6)$$

Now we show that both $\mathcal{H}_{\infty} x^m$ and $\mathcal{H}_{\infty} x^{m-1}$ are well-defined for all $x \in l^1$.

Proposition 1.1. *Let \mathcal{H}_{∞} be an m -order infinite dimensional generalized Hilbert tensor. Then for all $x \in l^1$,*

- (i) $\mathcal{H}_{\infty} x^m$ defined by (1.6) absolutely converges, i.e., $|\mathcal{H}_{\infty} x^m| < \infty$;
- (ii) $\mathcal{H}_{\infty} x^{m-1}$ is well-defined, i.e., for each positive integer i , its i th component defined by (1.5) absolutely converges.

Proof. Let $[a]$ denote the largest integer not exceeding a . Then for all $a \in \mathbb{R} \setminus \mathbb{Z}^-$, it is obvious that for all positive integers i_1, i_2, \dots, i_m ,

$$\min_{i_1, \dots, i_m} |i_1 + i_2 + \cdots + i_m - m + a| = a \text{ for } a > 0.$$

For $a < 0$, there exist some positive integers i'_1, i'_2, \dots, i'_m and $i''_1, i''_2, \dots, i''_m$ such that

$$i'_1 + i'_2 + \cdots + i'_m - m = -[a] \text{ and } i''_1 + i''_2 + \cdots + i''_m - m = -[a] - 1,$$

and hence,

$$\min_{i_1, \dots, i_m} |i_1 + i_2 + \cdots + i_m - m + a| = \min\{a - [a], 1 - (a - [a])\} \text{ for } a < 0.$$

In conclusion, we have for $a \in \mathbb{R} \setminus \mathbb{Z}^-$,

$$\frac{1}{|i_1 + i_2 + \cdots + i_m - m + a|} \leq N(a) = \begin{cases} \frac{1}{a}, & a > 0; \\ \frac{1}{\min\{a - [a], 1 + [a] - a\}}, & a < 0. \end{cases}$$

Then for $x = (x_1, x_2, \dots, x_n, x_{n+1}, \dots) \in l^1$, we obtain

$$\begin{aligned} |\mathcal{H}_\infty x^m| &= \left| \sum_{i_1, i_2, \dots, i_m=1}^{\infty} \frac{x_{i_1} x_{i_2} \cdots x_{i_m}}{i_1 + i_2 + \cdots + i_m - m + a} \right| \\ &\leq \sum_{i_1, i_2, \dots, i_m=1}^{\infty} \frac{|x_{i_1} x_{i_2} \cdots x_{i_m}|}{|i_1 + i_2 + \cdots + i_m - m + a|} \\ &\leq N(a) \sum_{i_1, i_2, \dots, i_m=1}^{\infty} |x_{i_1}| |x_{i_2}| \cdots |x_{i_m}| \\ &= N(a) \left(\sum_{i=1}^{\infty} |x_i| \right)^m < \infty \end{aligned}$$

since $\sum_{i=1}^{\infty} |x_i| < \infty$, and hence, $\mathcal{H}_\infty x^m$ absolutely converges.

Similarly, for each positive integer i , we also have

$$\begin{aligned} |(\mathcal{H}_\infty x^{m-1})_i| &= \left| \sum_{i_2, \dots, i_m=1}^{\infty} \frac{x_{i_2} \cdots x_{i_m}}{i + i_2 + \cdots + i_m - m + a} \right| \\ &\leq N(a) \left(\sum_{k=1}^{\infty} |x_k| \right)^{m-1} < \infty. \end{aligned}$$

So $\mathcal{H}_\infty x^{m-1}$ is well-defined. □

In section 2, we present some concepts and basic facts which are used for late. In section 3, we first show that both infinite and finite dimensional generalized Hilbert tensors are positive definite for $a \geq 1$. Let

$$F_\infty x = (\mathcal{H}_\infty x^{m-1})^{[\frac{1}{m-1}]} \text{ and } T_\infty x = \begin{cases} \|x\|_{l^1}^{2-m} \mathcal{H}_\infty x^{m-1} & x \neq \theta \\ \theta & x = \theta, \end{cases} \quad (1.7)$$

where $\theta = (0, 0, \dots, 0, \dots)^\top$. Then we prove that both F_∞ and T_∞ is a bounded, continuous and positively homogeneous operator on l^1 . In particular, their upper bounds with respect to operator norm are obtained as follows:

$$\|F_\infty\| = \sup_{\|x\|_{l^1}=1} \|F_\infty x\|_{l^{2(m-1)}} \leq K(a) \text{ and } \|T_\infty\| = \sup_{\|x\|_{l^1}=1} \|T_\infty x\|_{l^2} \leq C(a), \quad (1.8)$$

where

$$K(a) = \begin{cases} 2^{(m-1)} \sqrt{\frac{1}{a^2} + \frac{\pi^2}{6}}, & 0 < a < 1; \\ 2^{(m-1)} \sqrt{\frac{\pi^2}{6}}, & a \geq 1; \end{cases} \text{ and } C(a) = \begin{cases} \sqrt{\frac{1}{a^2} + \frac{\pi^2}{6}}, & 0 < a < 1; \\ \frac{\pi}{\sqrt{6}}, & a \geq 1. \end{cases}$$

For a finite dimensional generalized Hilbert tensor \mathcal{H}_n , we show that its H -spectral radius and its Z -spectral radius are smaller than or equal to $M(a)n^{m-1}$ and $M(a)n^{\frac{m}{2}}$, respectively,

where

$$M(a) = \begin{cases} \frac{1}{a}, & a > 0; \\ \frac{1}{\min\{a-[a], 1+[a]-a\}}, & -m(n-1) < a < 0; \\ \frac{1}{-m(n-1)-a}, & a < -m(n-1). \end{cases}$$

In particular, for $a > 0$, we obtain that the upper bounds of its spectral radius $\rho(\mathcal{H}_n)$ and its E -spectral radius $\rho_E(\mathcal{H}_n)$, i.e.,

$$\rho(\mathcal{H}_n) \leq \frac{n^{m-1}}{a} \text{ and } \rho_E(\mathcal{H}_n) \leq \frac{n^{\frac{m}{2}}}{a}. \quad (1.9)$$

2 Preliminaries and basic facts

Let both X and Y be two real Banach spaces with the norm $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and let $T : X \rightarrow Y$ be an operator. Then T is said to be

- (i) t -homogeneous if $T(\lambda x) = \lambda^t T(x)$ for all $\lambda \in \mathbb{R}$ and all $x \in X$;
- (ii) positively homogeneous if $T(tx) = tT(x)$ for all $t > 0$ and all $x \in X$;
- (iii) bounded if there exists a real number $M > 0$ such that

$$\|Tx\|_Y \leq M\|x\|_X \text{ for all } x \in X.$$

For a bounded, continuous and positively homogeneous operator $T : X \rightarrow Y$, the norm of T may be defined as follows (see Fucik et al. [6], Song and Qi [22] for more details):

$$\|T\| = \sup\{\|Tx\|_Y : \|x\|_X = 1\}. \quad (2.1)$$

For $0 < p < \infty$, l^p is a space consisting of all real number sequences $x = (x_i)_{i=1}^{\infty}$ satisfying $\sum_{i=1}^{\infty} |x_{i_i}|^p < \infty$. For $p \geq 1$, it is well known that

$$\|x\|_{l^p} = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}$$

is the norms defined on the sequences space l^p . For $p \geq 1$, it is well known that

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

is the norms defined on \mathbb{R}^n . Moreover, the following relationship between the two different norms is obvious,

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q \text{ for } q > p. \quad (2.2)$$

Let \mathcal{A} be an m -order n -dimensional symmetric tensor. Then a number λ is called an eigenvalue of \mathcal{A} if there exists a nonzero vector x such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \quad (2.3)$$

where $x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \dots, x_n^{m-1})^\top$, and call x an eigenvector of \mathcal{A} associated with the eigenvalue λ . We call such an eigenvalue H -eigenvalue if it is real and has a real eigenvector x , and call such a real eigenvector x an H -eigenvector. These concepts were first introduced by Qi [16] for the higher order symmetric tensors. Lim [14] independently introduced the notion of eigenvalue for higher order tensors but restricted x to be a real vector and λ to be a real number.

Qi [16] introduced another concept of tensor eigenvalue. A number μ is said to be an E -eigenvalue of \mathcal{A} if there exists a nonzero vector x such that

$$\mathcal{A}x^{m-1} = \mu x(x^\top x)^{\frac{m-2}{2}}, \quad (2.4)$$

and such a nonzero vector x is called an E -eigenvector of \mathcal{A} associated with μ . It is clear that if x is real, then μ is also real. In this case, μ and x are called a Z -eigenvalue of \mathcal{A} and a Z -eigenvector of \mathcal{A} , respectively. Qi [16, 17] extended some nice properties of symmetric matrices to higher order symmetric tensors. The Perron-Frobenius theorem of nonnegative matrices had been generalized to higher order nonnegative tensors under various conditions by Chang, Pearson and Zhang [1, 2], Qi [18], Song and Qi [21, 22], Yang and Yang [27, 28] and references therein.

3 Generalized Hilbert tensors

In this section, we mainly discuss the properties of infinite and finite dimensional generalized Hilbert tensors. It is easy to see that both finite and infinite dimensional generalized Hilbert tensor \mathcal{H}_n and \mathcal{H}_∞ are symmetric but not always positive semi-definite since $a \in \mathbb{R} \setminus \mathbb{Z}^-$. However, for all $a > 0$, both finite and infinite dimensional generalized Hilbert tensors are positive, it follows from the definition of strictly copositive tensors that \mathcal{H}_n and \mathcal{H}_∞ are strictly copositive, i.e.,

$$\mathcal{H}_n x^m > 0 \text{ for all nonnegative nonzero vector } x \in \mathbb{R}^n,$$

$$\mathcal{H}_\infty x^m > 0 \text{ for all nonnegative nonzero vector } x \in l^1.$$

The concept of strictly copositive tensors was introduced by Qi [18]. Also see Song and Qi [23] for more details. Now, we give the positive definiteness of such two tensors when $a \geq 1$.

3.1 Positive definiteness

Theorem 3.1. *Let m be even and $a \geq 1$. Then m -order generalized Hilbert tensors \mathcal{H}_∞ and \mathcal{H}_n are both positive definite, i.e.,*

$$\mathcal{H}_\infty x^m > 0 \text{ for all nonzero vector } x \in l^1,$$

$$\mathcal{H}_n x^m > 0 \text{ for all nonzero vector } x \in \mathbb{R}^n.$$

Proof. For each $x = (x_1, x_2, \dots, x_n, x_{n+1}, \dots) \in l^1$ and $t \in [0, 1]$, it is obvious that

$$|t^{i-1+\frac{a-1}{m}} x_i| = |t|^{i-1+\frac{a-1}{m}} |x_i| \leq |x_i|,$$

which implies that the infinite series $\sum_{i=1}^{\infty} t^{i-1+\frac{a-1}{m}} x_i$ uniformly converges with respect to $t \in [0, 1]$. Then it follows from integral form of $\mathcal{H}_{i_1 i_2 \dots i_m}$ that

$$\begin{aligned} \mathcal{H}_{\infty} x^m &= \sum_{i_2, i_2, \dots, i_m=1}^{\infty} \int_0^1 t^{i_1+i_2+\dots+i_m-m+a-1} x_{i_1} x_{i_2} \dots x_{i_m} dt \\ &= \sum_{i_1, i_2, \dots, i_m=1}^{\infty} \int_0^1 (t^{i_1-1+\frac{a-1}{m}} x_{i_1}) (t^{i_2-1+\frac{a-1}{m}} x_{i_2}) \dots (t^{i_m-1+\frac{a-1}{m}} x_{i_m}) dt \\ &= \int_0^1 \left(\sum_{i_1=1}^{\infty} t^{i_1-1+\frac{a-1}{m}} x_{i_1} \right) \left(\sum_{i_2=1}^{\infty} t^{i_2-1+\frac{a-1}{m}} x_{i_2} \right) \dots \left(\sum_{i_m=1}^{\infty} t^{i_m-1+\frac{a-1}{m}} x_{i_m} \right) dt \\ &= \int_0^1 \left(\sum_{i=1}^{\infty} t^{i-1+\frac{a-1}{m}} x_i \right)^m dt \geq 0 \quad (m \text{ is even}). \end{aligned}$$

Now we show $\mathcal{H}_{\infty} x^m > 0$ for all nonzero vector $x \in l^1$. Suppose not, then there exists a nonzero vector $x \in l^1$ such that $\mathcal{H}_{\infty} x^m = 0$, i.e.,

$$\int_0^1 \left(\sum_{i=1}^{\infty} t^{i-1+\frac{a-1}{m}} x_i \right)^m dt = 0.$$

Since m is even, $\left(\sum_{i=1}^{\infty} t^{i-1+\frac{a-1}{m}} x_i \right)^m \geq 0$, and hence, by the continuity, we have

$$t^{\frac{a-1}{m}} \left(x_1 + \left(\sum_{i=2}^{\infty} t^{i-1} x_i \right) \right) = \sum_{i=1}^{\infty} t^{i-1+\frac{a-1}{m}} x_i = 0 \text{ for all } t \in [0, 1].$$

So for all $t \in (0, 1]$, we have

$$x_1 + \left(\sum_{i=2}^{\infty} t^{i-1} x_i \right) = 0.$$

Since the polynomial $x_1 + \left(\sum_{i=2}^{\infty} t^{i-1} x_i \right)$ is continuous at $t = 0$, we see that

$$x_1 + \left(\sum_{i=2}^{\infty} t^{i-1} x_i \right) = 0 \text{ for all } t \in [0, 1].$$

Let $t = 0$. Then $x_1 = 0$, and so,

$$t^{\frac{a-1}{m}+1} \left(x_2 + \sum_{i=3}^{\infty} t^{i-2} x_i \right) = \sum_{i=2}^{\infty} t^{i-1+\frac{a-1}{m}} x_i = 0 \text{ for all } t \in [0, 1].$$

So, for all $t \in (0, 1]$, we have

$$x_2 + \sum_{i=3}^{\infty} t^{i-2} x_i = 0.$$

Again by the continuity, we see that

$$x_2 + \sum_{i=3}^{\infty} t^{i-2} x_i = 0 \text{ for all } t \in [0, 1].$$

Take $t = 0$, then we have $x_2 = 0$.

By the mathematical induction, suppose $x_1 = x_2 = \cdots = x_k = 0$. Then we only need show $x_{k+1} = 0$. In fact,

$$t^{\frac{a-1}{m}+k} \left(x_{k+1} + \left(\sum_{i=k+2}^{\infty} t^{i-(k+1)} x_i \right) \right) = \sum_{i=k+1}^{\infty} t^{i-1+\frac{a-1}{m}} x_i = 0 \text{ for all } t \in [0, 1].$$

So, for all $t \in (0, 1]$, we have

$$x_{k+1} + \left(\sum_{i=k+2}^{\infty} t^{i-(k+1)} x_i \right) = 0.$$

Again by the continuity, we see that

$$x_{k+1} + \left(\sum_{i=k+2}^{\infty} t^{i-(k+1)} x_i \right) = 0 \text{ for all } t \in [0, 1].$$

Take $t = 0$, then we have $x_{k+1} = 0$. So we may conclude that

$$x_i = 0 \text{ for all } i = 1, 2, \dots, n, n+1, \dots,$$

and hence, $x = \theta = (0, 0, \dots, 0, 0, \dots)^\top$, which is a contradiction. The desired conclusion follows.

Similarly, we can obtain that \mathcal{H}_n is positive definite. \square

Remark 3.1. We show the positive definiteness of infinite and finite dimensional generalized Hilbert tensors with $a \geq 1$. Then it is unknown whether or not \mathcal{H}_n and \mathcal{H}_∞ have the positive definiteness for $0 < a < 1$ or $a < 0$ with $a \in \mathbb{R} \setminus \mathbb{Z}^-$.

3.2 Infinite dimensional generalized Hilbert tensors

Let

$$F_\infty x = (\mathcal{H}_\infty x^{m-1})^{[\frac{1}{m-1}]} \quad (m \text{ is even}) \quad (3.1)$$

and

$$T_\infty x = \begin{cases} \|x\|_{l^1}^{2-m} \mathcal{H}_\infty x^{m-1} & x \neq \theta \\ \theta & x = \theta, \end{cases} \quad (3.2)$$

where $x^{[\frac{1}{m-1}]} = (x_1^{\frac{1}{m-1}}, x_2^{\frac{1}{m-1}}, \dots, x_n^{\frac{1}{m-1}}, \dots)^\top$ and $\theta = (0, 0, \dots, 0, \dots)^\top$, the zero element of a vector space l^p . It is easy to see that both operators F_∞ and T_∞ are continuous and positively homogeneous. Therefore our main interests are to study the boundedness of two classes of operators.

Theorem 3.2. Let F_∞ and T_∞ are defined by Eqs. (3.1) and (3.2), respectively. Assume that $a > 0$.

(i) If $x \in l^1$, then $F_\infty x \in l^p$ for $m-1 < p < \infty$. Moreover, F_∞ is a bounded, continuous and positively homogeneous operator from l^1 into l^p ($m-1 < p < \infty$). In particular,

$$\|F_\infty\| = \sup_{\|x\|_{l^1}=1} \|F_\infty x\|_{l^{2(m-1)}} \leq K(a),$$

where

$$K(a) = \begin{cases} 2^{(m-1)} \sqrt{\frac{1}{a^2} + \frac{\pi^2}{6}}, & 0 < a < 1; \\ 2^{(m-1)} \sqrt{\frac{\pi^2}{6}}, & a \geq 1. \end{cases}$$

(ii) If $x \in l^1$, then $T_\infty x \in l^p$ for $1 < p < \infty$. Moreover, T_∞ is a bounded, continuous and positively homogeneous operator from l^1 into l^p ($1 < p < \infty$). In particular,

$$\|T_\infty\| = \sup_{\|x\|_{l^1}=1} \|T_\infty x\|_{l^2} \leq C(a),$$

where

$$C(a) = \begin{cases} \sqrt{\frac{1}{a^2} + \frac{\pi^2}{6}}, & 0 < a < 1; \\ \frac{\pi}{\sqrt{6}}, & a \geq 1. \end{cases}$$

Proof. For $x \in l^1$,

$$\begin{aligned} |(\mathcal{H}_\infty x^{m-1})_i| &= \lim_{n \rightarrow \infty} \left| \sum_{i_2, \dots, i_m=1}^n \frac{x_{i_2} \cdots x_{i_m}}{i + i_2 + \cdots + i_m - m + a} \right| \\ &\leq \lim_{n \rightarrow \infty} \sum_{i_2, \dots, i_m=1}^n \frac{|x_{i_2} \cdots x_{i_m}|}{|i + i_2 + \cdots + i_m - m + a|} \\ &\leq \lim_{n \rightarrow \infty} \sum_{i_2, \dots, i_m=1}^n \frac{|x_{i_2}| |x_{i_3}| \cdots |x_{i_m}|}{|i + \underbrace{1 + \cdots + 1}_{m-1} - m + a|} \\ &= \frac{1}{i - 1 + a} \lim_{n \rightarrow \infty} \sum_{i_2, \dots, i_m=1}^n |x_{i_2}| |x_{i_3}| \cdots |x_{i_m}| \\ &= \frac{1}{i - 1 + a} \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n |x_k| \right)^{m-1} \\ &= \frac{1}{i - 1 + a} \left(\sum_{k=1}^\infty |x_k| \right)^{m-1} \\ &= \frac{1}{i - 1 + a} \|x\|_{l^1}^{m-1}. \end{aligned}$$

Since $a > 0$, then for all positive integer $i > 1$, we have

$$\frac{1}{(i - 1 + a)^s} \leq \frac{1}{(i - 1)^s}.$$

So the series $\sum_{i=1}^{\infty} \frac{1}{(i-1+a)^s}$ converges whenever $s > 1$.

(i) For $m-1 < p < \infty$, it follows from the definition of F_{∞} that

$$\begin{aligned} \sum_{i=1}^{\infty} |(F_{\infty}x)_i|^p &= \sum_{i=1}^{\infty} |(\mathcal{H}_{\infty}x^{m-1})_i^{\frac{1}{m-1}}|^p \\ &= \sum_{i=1}^{\infty} |(\mathcal{H}_{\infty}x^{m-1})_i|^{\frac{p}{m-1}} \\ &\leq \sum_{i=1}^{\infty} \left(\frac{1}{(i-1+a)} \|x\|_{l^1}^{m-1} \right)^{\frac{p}{m-1}} \\ &= \|x\|_{l^1}^p \sum_{i=1}^{\infty} \frac{1}{(i-1+a)^{\frac{p}{m-1}}} < \infty \end{aligned}$$

since $s = \frac{p}{m-1} > 1$. Thus $F_{\infty}x \in l^p$ for all $x \in l^1$. Furthermore, we have

$$\|F_{\infty}x\|_{l^p} = \left(\sum_{i=1}^{\infty} |(F_{\infty}x)_i|^p \right)^{\frac{1}{p}} \leq M \|x\|_{l^1}, \quad (3.3)$$

where $M = \left(\sum_{i=1}^{\infty} \frac{1}{(i-1+a)^{\frac{p}{m-1}}} \right)^{\frac{1}{p}}$. Therefore F_{∞} is a bounded, continuous and positively homogeneous operator from l^1 into l^p ($m-1 < p < \infty$). Take $p = 2(m-1)$, then for $a \geq 1$, we have $a-1 \geq 0$, and hence,

$$M = \left(\sum_{i=1}^{\infty} \frac{1}{(i-1+a)^2} \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} \frac{1}{i^2} \right)^{\frac{1}{p}} = {}^{2(m-1)}\sqrt{\frac{\pi^2}{6}}.$$

If $0 < a < 1$, then

$$M = \left(\sum_{i=1}^{\infty} \frac{1}{(i-1+a)^2} \right)^{\frac{1}{p}} \leq \left(\frac{1}{(1-1+a)^2} + \sum_{i=2}^{\infty} \frac{1}{(i-1)^2} \right)^{\frac{1}{p}} = {}^{2(m-1)}\sqrt{\frac{1}{a^2} + \frac{\pi^2}{6}}.$$

It follows from (2.1) and (3.3) that

$$\|F_{\infty}\| = \sup_{\|x\|_{l^1}=1} \|F_{\infty}x\|_{l^{2(m-1)}} \leq M \leq K(a).$$

(ii) For $1 < p < \infty$, it follows from the definition of T_{∞} that

$$\begin{aligned} \sum_{i=1}^{\infty} |(T_{\infty}x)_i|^p &= \sum_{i=1}^{\infty} |(\|x\|_{l^1}^{2-m} \mathcal{H}_{\infty}x^{m-1})_i|^p \\ &= \|x\|_{l^1}^{(2-m)p} \sum_{i=1}^{\infty} |(\mathcal{H}_{\infty}x^{m-1})_i|^p \\ &\leq \|x\|_{l^1}^{(2-m)p} \sum_{i=1}^{\infty} \left(\frac{1}{(i-1+a)} \|x\|_{l^1}^{m-1} \right)^p \\ &= \|x\|_{l^1}^p \sum_{i=1}^{\infty} \frac{1}{(i-1+a)^p} < \infty \end{aligned}$$

since $s = p > 1$. Thus, $T_\infty x \in l^p$ for all $x \in l^1$, Furthermore, we have

$$\|T_\infty x\|_{l^p} = \left(\sum_{i=1}^{\infty} |(T_\infty x)_i|^p \right)^{\frac{1}{p}} \leq C \|x\|_{l^1}, \quad (3.4)$$

where $C = \left(\sum_{i=1}^{\infty} \frac{1}{(i-1+a)^p} \right)^{\frac{1}{p}} > 0$. So T_∞ is a bounded, continuous and positively homogeneous operator from l^1 into l^p ($1 < p < \infty$). Similarly, take $p = 2$, then for $a \geq 1$, we have

$$C = \left(\sum_{i=1}^{\infty} \frac{1}{(i-1+a)^2} \right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{\infty} \frac{1}{i^2} \right)^{\frac{1}{2}} \leq \sqrt{\frac{\pi^2}{6}}.$$

If $0 < a < 1$, then

$$C = \left(\sum_{i=1}^{\infty} \frac{1}{(i-1+a)^2} \right)^{\frac{1}{2}} \leq \left(\frac{1}{(1-1+a)^2} + \sum_{i=2}^{\infty} \frac{1}{(i-1)^2} \right)^{\frac{1}{2}} = \sqrt{\frac{1}{a^2} + \frac{\pi^2}{6}}.$$

It follows from (2.1) and (3.4) that

$$\|T_\infty\| = \sup_{\|x\|_{l^1}=1} \|T_\infty x\|_{l^2} \leq C \leq C(a).$$

The desired results follow. \square

From the proof of Theorem 3.2, it is easy to obtain the following properties of operator defined by infinite dimensional generalized Hilbert tensor \mathcal{H}_∞ .

Theorem 3.3. *For an m -order infinite dimensional generalized Hilbert tensor \mathcal{H}_∞ and $a > 0$, let $f(x) = \mathcal{H}_\infty x^{m-1}$. Then f is a bounded, continuous and positively $(m-1)$ -homogeneous operator from l^1 into l^p ($1 < p < \infty$).*

Remark 3.2. (i) *In Theorem 3.3, \mathcal{H}_∞ defines a bounded operator for $a > 0$. Then it is unknown whether or not \mathcal{H}_∞ is bounded for $a < 0$ with $a \in \mathbb{R} \setminus \mathbb{Z}^-$.*

(ii) *For an m -order infinite dimensional generalized Hilbert tensor, the upper bounds of norm of corresponding positively homogeneous operators are showed for $a > 0$, then for $a < 0$ with $a \in \mathbb{R} \setminus \mathbb{Z}^-$, it is unknown whether have similar conclusions or not.*

(iii) *Are the upper bounds the best in Theorem 3.2?*

3.3 Finite dimensional generalized Hilbert tensors

Theorem 3.4. *Let \mathcal{H}_n be an m -order n -dimensional generalized Hilbert tensor. Assume that $a \in \mathbb{R} \setminus \mathbb{Z}^-$ and*

$$M(a) = \begin{cases} \frac{1}{a}, & a > 0; \\ \frac{1}{\min\{a-[a], 1+[a]-a\}}, & -m(n-1) < a < 0; \\ \frac{1}{-m(n-1)-a}, & a < -m(n-1), \end{cases}$$

where $[a]$ denotes the largest integer not exceeding a . Then

(i) $|\lambda| \leq M(a)n^{m-1}$ for all H -eigenvalues λ of generalized Hilbert tensor \mathcal{H}_n if m is even;

(ii) $|\mu| \leq M(a)n^{\frac{m}{2}}$ for all Z -eigenvalues μ of generalized Hilbert tensor \mathcal{H}_n .

Proof. Since $a \in \mathbb{R} \setminus \mathbb{Z}^-$, Using the proof technique of Proposition 1.1, it is obvious that for all $i_1, i_2, \dots, i_m \in \{1, 2, \dots, n\}$,

$$\begin{aligned} |i_1 + i_2 + \dots + i_m - m + a| &\geq a \text{ for } a > 0, \\ |i_1 + i_2 + \dots + i_m - m + a| &\geq \min\{a - [a], 1 - (a - [a])\} \text{ for } -m(n-1) < a < 0, \\ |i_1 + i_2 + \dots + i_m - m + a| &\geq -m(n-1) - a \text{ for } a < -m(n-1). \end{aligned}$$

In conclusion,

$$\frac{1}{|i_1 + i_2 + \dots + i_m - m + a|} \leq M(a).$$

Then for each nonzero vector $x \in \mathbb{R}^n$, we have

$$\begin{aligned} |\mathcal{H}_n x^m| &= \left| \sum_{i_1, i_2, \dots, i_m=1}^n \frac{x_{i_1} x_{i_2} \dots x_{i_m}}{i_1 + i_2 + \dots + i_m - m + a} \right| \\ &\leq \sum_{i_1, i_2, \dots, i_m=1}^n \frac{|x_{i_1} x_{i_2} \dots x_{i_m}|}{|i_1 + i_2 + \dots + i_m - m + a|} \\ &\leq M(a) \sum_{i_1, i_2, \dots, i_m=1}^n |x_{i_1}| |x_{i_2}| \dots |x_{i_m}| \\ &= M(a) \left(\sum_{i=1}^n |x_i| \right)^m \\ &= M(a) \|x\|_1^m. \end{aligned}$$

(i) From the fact that $\|x\|_1 \leq n^{1-\frac{1}{m}} \|x\|_m$, it follows that

$$\begin{aligned} |\mathcal{H}_n x^m| &\leq M(a) \|x\|_1^m \leq M(a) (n^{1-\frac{1}{m}} \|x\|_m)^m \\ &= M(a) n^{m-1} \|x\|_m^m, \end{aligned}$$

and hence, for all nonzero vector $x \in \mathbb{R}^n$, we have

$$\left| \mathcal{H}_n \left(\frac{x}{\|x\|_m} \right)^m \right| = \frac{|\mathcal{H}_n x^m|}{\|x\|_m^m} \leq M(a) n^{m-1}. \quad (3.5)$$

It follows from the definition of eigenvalues of tensor that for each H -eigenvalue λ of generalized Hilbert tensor \mathcal{H}_n , there exists a nonzero vector $y \in \mathbb{R}^n$ such that

$$\mathcal{H}_n y^{m-1} = \lambda y^{[m-1]},$$

and so,

$$\mathcal{H}_n y^m = y^\top (\mathcal{H}_n y^{m-1}) = \lambda y^\top y^{[m-1]} = \lambda \sum_{i=1}^n y_i^m = \lambda \|y\|_m^m$$

since m is even. Thus, it follows from (3.5) that

$$|\lambda| = \frac{|\mathcal{H}_n y^m|}{\|y\|_m^m} = \left| \mathcal{H}_n \left(\frac{y}{\|y\|_m} \right)^m \right| \leq M(a) n^{m-1}.$$

Since λ is arbitrary H -eigenvalue, this obtains the conclusion (i).

(ii) Similarly, from the fact that $\|x\|_1 \leq \sqrt{n}\|x\|_2$, it follows that

$$|\mathcal{H}_n x^m| \leq M(a) \|x\|_1^m \leq M(a) n^{\frac{m}{2}} \|x\|_2^m,$$

and hence, for all nonzero vector $x \in \mathbb{R}^n$, we have

$$\left| \mathcal{H}_n \left(\frac{x}{\|x\|_2} \right)^m \right| = \frac{|\mathcal{H}_n x^m|}{\|x\|_2^m} \leq M(a) n^{\frac{m}{2}}. \quad (3.6)$$

It follows from the definition of Z -eigenvalues of tensor that for each Z -eigenvalue μ of generalized Hilbert tensor \mathcal{H}_n , there exists a nonzero vector $z \in \mathbb{R}^n$ such that

$$\mathcal{H}_n z^{m-1} = \mu z (z^\top z)^{\frac{m-2}{2}},$$

and so,

$$\mathcal{H}_n z^m = z^\top (\mathcal{H}_n z^{m-1}) = \mu z^\top z (z^\top z)^{\frac{m-2}{2}} = \mu \left(\sum_{i=1}^n z_i^2 \right)^{\frac{m}{2}} = \mu \|z\|_2^m.$$

Thus, by (3.6), we have

$$|\mu| = \frac{|\mathcal{H}_n z^m|}{\|z\|_2^m} = \left| \mathcal{H}_n \left(\frac{z}{\|z\|_2} \right)^m \right| \leq M(a) n^{\frac{m}{2}}.$$

Since μ is arbitrary Z -eigenvalue, the desired conclusion follows. \square

Let $a > 0$. Then the generalized Hilbert tensor \mathcal{H}_n is positive (all entries are positive) and symmetric, and hence, its spectral radius $\rho(\mathcal{H}_n)$ and its E -spectral radius $\rho_E(\mathcal{H}_n)$ is its H -eigenvalue and its Z -eigenvalue, respectively. For more details, see Chang, Pearson and Zhang [1, 2], Qi [18], Song and Qi [21, 22] and Yang and Yang [27, 28]. So the following colusions are easy to obtain by Theorem 3.4.

Corollary 3.5. *Let \mathcal{H}_n be an m -order n -dimensional generalized Hilbert tensor. Assume that $a > 0$. Then*

(i) $|\lambda| \leq \rho(\mathcal{H}_n) \leq \frac{n^{m-1}}{a}$ for all eigenvalues λ of generalized Hilbert tensor \mathcal{H}_n if m is even;

(ii) $|\mu| \leq \rho_E(\mathcal{H}_n) \leq \frac{n^{\frac{m}{2}}}{a}$ for all E -eigenvalues μ of generalized Hilbert tensor \mathcal{H}_n .

Remark 3.3. *We prove the upper bounds of spectral radii in the above results. These upper bounds may not be the best, then the following problems deserve us further research:*

What is the best upper bounds? How compute such an upper bound?

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